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158. Proposed by G. H. HARVILL, A. M., Malakoff, Texas.

Show that a law of density for points in space may be assumed such that the joint mass of any two points which are electrical images of each other in respect to a given sphere may be constant, and that their centers of gravity shall lie on the surface of the sphere.

Remark by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

This problem is the same as problem 148, a solution of which is given in No. 2, Vol. X, pages 48 and 49.

### DIOPHANTINE ANALYSIS.

105. Proposed by HARRY S. VANDIVER, Bala, Pa.

Prove that every factor of  $a^{2m} + b^{2m}$  is of the form  $1[\text{mod}.2^{m+1}]$  where  $a$  and  $b$  are prime to each other.

Note by the PROPOSER.

The enunciation of this proposition is corrected on page 207, Vol. IX, of the MONTHLY, and it may be stated in full as follows:

*If  $a$ ,  $b$  and  $m$  are real positive integers,  $a$  and  $b$  relatively prime, then every odd factor of  $a^{2m} + b^{2m}$  has the form  $1[\text{mod}.2^{m+1}]$ .*

This special arithmetical theorem was first given by Euler, and may be directly proved by the principles in the theory of power residues. It is interesting, however, to examine its connection with a more general proposition. If  $\omega_1, \omega_2, \dots, \omega_{\phi(n)}$  are all the distinct primitive roots of  $x^n - 1 = 0$  and  $\phi(n)$  the number of integers less than the positive integer  $n$ , and prime to it, then the rational function

$$F\{a, b, n\} = \{a - \omega_1 b\} \{a - \omega_2 b\} \dots \{a - \omega_{\phi(n)} b\}$$

possesses the following property:

*Every divisor of  $F\{a, b, n\}$  which is prime to  $n$ , has the form  $1[\text{mod } n]$ .*

A proof of this for the case  $b=1$ , is to be found in Kronecker's *Vorlesungen über Zahlentheorie*, Vol. I, pages 440-441, and the analysis used may be readily extended to cover the case in which  $b$  is general. For  $n=2^{m+1}$  ( $m$ , a positive integer) it will be noticed that

$$F\{a, b, 2^{m+1}\} = a^{2m} + b^{2m},$$

and by the general theorem above stated, it follows that every odd divisor of this function has the form  $1[\text{mod } 2^{m+1}]$ . When  $a=2$ ,  $b=1$ , we may write

$$f\{m\} = 2^{2m} + 1,$$

and this form gives the remarkable numbers of Gauss which occur in the partition of the perigon.

I notice that the determination of factors of  $f\{m\}$  for  $m=9, 11, 12, 18$  and  $38$ , has been recently effected by several English mathematicians. For information regarding these interesting results, see the *Educational Times*, June, 1903, page 270, where reference is made to the announcement by Col. J. A. Cunningham. A summary of the previous work done along this line is given in Ball's *Recreations and Problems*, third edition, pages 34-35.

112. Proposed by L. C. WALKER, A. M., Graduate Student Leland Stanford Jr. University, Cal.

There is a series of rational triangles whose sides have a common difference of unity. Calling the one whose sides are 3, 4, 5, the first triangle, find the sides of the  $n$ th triangle.

I. Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and the PROPOSER.

Let  $2x_n-1$ ,  $2x_n$ , and  $2x_n+1$  denote the sides of the  $n$ th triangle,  $r_n$  the radius of the inscribed circle, and  $A$  the area of the triangle.

$$\text{Then } A = \sqrt{\{3x_n(x_n+1)(x_n-1)\}} = 3x_nr_n,$$

$$\text{whence } [x_n - r_n\sqrt{3}][x_n + r_n\sqrt{3}] = 1 \dots [1].$$

$$\text{Now for the first triangle, } x=2, \text{ and } r=1; \therefore [2 - \sqrt{3}]^n [2 + \sqrt{3}]^n = 1 \dots [2].$$

Assume  $x_n - r_n\sqrt{3} = [2 - \sqrt{3}]^n$ ; then from [1],  $x_n + r_n\sqrt{3} = [2 + \sqrt{3}]^n$ ; from which  $x_n - r_n\sqrt{3} = \frac{1}{2}\{[2 + \sqrt{3}]^n + [2 - \sqrt{3}]^n\}$ . Hence,

$$2x_n - 1 = [2 + \sqrt{3}]^n + [2 - \sqrt{3}]^n - 1,$$

$$2x_n = [2 + \sqrt{3}]^n + [2 - \sqrt{3}]^n,$$

$$2x_n + 1 = [2 + \sqrt{3}]^n + [2 - \sqrt{3}]^n + 1,$$

$$A = [\frac{1}{4}\sqrt{3}]\{[2 + \sqrt{3}]^{2n} - [2 - \sqrt{3}]^{2n}\}.$$

II. Solution by A. H. BELL, Litchfield, Ill.

Take the sides of the rational triangle  $x+d$ ,  $x$ ,  $x-d$ ; the area of this triangle is  $\frac{1}{2}x[3(x^2-4d^2)]^{\frac{1}{2}}$ . To have it rational we must have  $3[x^2-4d^2] = \square = y^2$ , say, in which the area is  $\frac{1}{2}xy$ . If we make  $x=2x'$ , and  $y=6y'$ , the equation becomes, when  $d=1$ ,  $x'^2-3y'^2=1 \dots (2)$ . The values of  $x$ , by Lagrange, are the numerators of the odd convergents from the  $\sqrt{3}$ ; the first two odd fractions are  $\frac{1}{3}$ ,  $\frac{2}{1}$ , etc., in which  $2x'=x=4$ , giving the initial rational triangle with sides 3, 4, 5.

The extension of the values of  $x'$ , or  $x$ , by the algebraic solution of this equation in general, is known to be  $x_{n+1}=2x_nx_{n-1} \dots (4)$ .

This constant factor  $2x$ , is called by Roberts, and Robins, the magic  $M$ , or Key. In this case  $M=4$ , and the middle side of the next rational triangle is by (4),  $x_2=4 \cdot 4 - 2 = 14$ , and the triangle 13, 14, 15; by extending the middle side in the same way we get,

|                  |   |    |    |     |     |      |              |
|------------------|---|----|----|-----|-----|------|--------------|
| No. of triangle: | 1 | 2  | 3  | 4   | 5   | 6    |              |
| Sides:           | 3 | 13 | 51 | 193 | 723 | 2701 |              |
|                  | 4 | 14 | 52 | 194 | 724 | 2702 | etc.....(5). |
|                  | 5 | 15 | 53 | 195 | 725 | 2703 |              |

The  $n$ th rational triangle can be found by extending formula (4) in terms of  $M$ , and  $x'$ , as follows: